Period 1, Sept 9, 2024

a) If $f(x) = \frac{1}{x}$, what is $f(c)$? $f(c+h)$?

$F(c) = \frac{1}{c}$

$F(c+h) = \frac{1}{c+h}$

b) If $f(x) = x^3 + 3x$, what is $f(c)$? $f(c+h)$?

$F(c) = c^3 + 3c$

$F(c+h) = (c+h)^3 + 3(c+h) = (c^3 + 3c^2h + 3ch^2 + h^3) + 3c + 3h$

c) If $f(x) = \sqrt{2x-2}$, what is $f(c)$? $f(c+\Delta x)$?

$F(c) = \sqrt{2(c)-2}$

$F(c+\Delta x) = \sqrt{2(c+\Delta x)-2} = \sqrt{2c+2\Delta x-2}$

d) How do you find the Slope between two points?

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{F(x_2) - F(x_1)}{x_2 - x_1} \Rightarrow m = \frac{F(x+h) - F(x)}{(x+h) - x} = \frac{F(x+h) - F(x)}{\cancel{x}+h-\cancel{x}}$

$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$ = Finding The Slope of 2 Points That are infinitly close

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Example 1: Find the slope of the graph of $f(x) = 2x - 3$ at the point (2, 1)

$c =$ ______

Average Rate of Change v.s. Instantaneous Rate of Change

$$\text{Average Rate of Change} = \frac{f(b) - f(a)}{b - a}$$

DEFINITION: Instantaneous Rate of Change

The **instantaneous rate of change** of *f* at *c* is the limit as *x* approaches *c* of the average rate of change. Symbolically, the instantaneous rate of change of *f* at *c* is

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Alternate Form of Instantaneous ROC

The **Instantaneous ROC** of *f* at a real number *c* has been defined as the real number $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$.

The alternative form of the **Instantaneous ROC** of f at a real number *c* is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

or

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

line J slope $= \frac{a}{b}$

$\perp$ Line J slope $= \frac{-b}{a}$

$\text{Line} \Rightarrow y = mx + b$

$F(8) = \sqrt{2 \cdot 8} = \sqrt{16} = 4$

$\text{Point}\ (8,4)\quad m = \frac{1}{4}$

Example 5:

$4 = 8 \cdot \frac{1}{4} + b \Rightarrow 4 = 2 + b \Rightarrow b = 2$

Tangent Line

$y = \frac{1}{4}x + 2$

(a) Find the int. rate of change of $f(x) = \sqrt{2x}$ at $x = 8$.

$c =$ ______

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \frac{\left(\sqrt{2(x+h)} - \sqrt{2x}\right)\left(\sqrt{2(x+h)} + \sqrt{2x}\right)}{h\left(\sqrt{2(x+h)} + \sqrt{2x}\right)}$$

$$\lim_{h \to 0} \frac{2(x+h) - \cancel{\sqrt{2x \cdot 2(x+h)}} + \cancel{\sqrt{2x \cdot 2(x+h)}} - 2x}{h\left(\sqrt{2(x+h)} + \sqrt{2x}\right)}$$

$$\lim_{h \to 0} \frac{\cancel{2x} + 2h - \cancel{2x}}{h\left(\sqrt{2(x+h)} + \sqrt{2x}\right)} = \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)} + \sqrt{2x}\right)}$$

$$\lim_{h \to 0} \frac{2}{\sqrt{2(x+h)} + \sqrt{2x}} = \frac{2}{\sqrt{2x} + \sqrt{2x}} = \frac{\cancel{2}}{\cancel{2}\sqrt{2x}}$$

$$\frac{d}{dx}(F(x)) = F'(x) = \frac{dy}{dx} = \frac{1}{\sqrt{2x}} \qquad x = 8 = \frac{1}{\sqrt{2 \cdot 8}} = \frac{1}{4}$$

Find Tangent and Normal lines as well.

Normal Line

$(8,4)\quad m = -\frac{4}{1} = -4$

$y = -4x + b$

$4 = -4(8) + b$

$4 = -32 + b$

$+36 = b \qquad y = -4x + 36$

$F(x) = \sqrt{2x}$

$$\lim_{x \to c} \frac{F(x) - F(c)}{x - c} = aT \qquad c = 8$$

$$\lim_{x \to 8} \frac{\sqrt{2x} - \sqrt{2 \cdot 8}}{x - 8} = \lim_{x \to 8} \frac{\sqrt{2x} - \sqrt{16}}{x - 8} = \lim_{x \to 8} \frac{\left(\sqrt{2x} - 4\right)\left(\sqrt{2x} + 4\right)}{(x - 8)\left(\sqrt{2x} + 4\right)}$$

$$\lim_{x \to 8} \frac{2x - \cancel{4\sqrt{2x}} + \cancel{4\sqrt{2x}} - 16}{(x-8)\left(\sqrt{2x} + 4\right)} = \lim_{x \to 8} \frac{2\cancel{(x-8)}}{\cancel{(x-8)}\left(\sqrt{2x} + 4\right)} = \lim_{x \to 8} \frac{2}{\sqrt{2x} + 4}$$

$$\frac{2}{\sqrt{2 \cdot 8} + 4} = \frac{2}{\sqrt{16} + 4}$$

$$\frac{2}{4 + 4} = \frac{2}{8} = \frac{1}{4}$$

$\frac{0}{0} = \emptyset$ $\quad x^3 - 2^3 \quad x^2 - 2^2$

$a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

20. $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)} = \frac{4+4+4}{2+2} = \frac{12}{4} = 3$

$x = 2$

A. 4 B. 0 C. 1 D. 3 E. 2

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24. If $f(x) = 3x^2 - 5x$, then find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

A. $3x - 5$
B. $6x - 5$
C. $6x$
D. 0
E. Does not exist

$F(x) = 3x^2 - 5x$

$F(x+h) = 3(x+h)^2 - 5(x+h)$

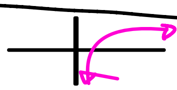
$= 3(x^2 + 2xh + h^2) - 5(x+h)$

$$\lim_{h \to 0} \frac{\overbrace{3(x^2+2xh+h^2) - 5(x+h)}^{F(x+h)} - \overbrace{[3x^2 - 5x]}^{F(x)}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{- 5x} - 5h \cancel{- 3x^2} \cancel{+ 5x}}{h} = \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 5)}{\cancel{h}}$$

$6x + 3 \cdot 0 - 5$

$6x - 5$



9. $\lim_{x \to 2^-} \ln(\underbrace{-x+2}) = -\infty = DNE$

1.9 going to zero $\ln(-1.9+2) = \ln(.1)$

Positive

$y = \ln_e x$

$e^y = x$

y can be anything

x is positive

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17. Find the values of k and m so that the function below is continuous on the interval $(-\infty, \infty)$.

$$f(x) = \begin{cases} x^2 - kx + 3, & x < -2 \\ 2x - 3, & -2 \le x \le 3 \\ 3 - 2m, & x > 3 \end{cases}$$

$\lim_{x \to c}$ must exist

$F(c)$ must exist

$\lim_{x \to c} F(x) = F(c)$

$-7 = (-2)^2 - k(-2) + 3$

$(-2)^2 - k(-2) + 3 = 2(-2) - 3$

$= -4 - 3$

$= -7$

$4 + 2k + 3 = -7$

$-4 \quad -3 \quad -4 - 3$

$2k = -14$

$k = -7$

$2(3) - 3 = 3 - 2m$

$6 - 3 = 3 - 2m$

$-3 \quad -3$

$0 = -2m$

$3 = 3 - 2m$

$0 = -2m$

$m = 0$

18. $\lim_{x\to 0} \frac{4x-3}{7x+1} = \frac{4(RSN)-3}{7(RSN)+1} = \frac{0-3}{0+1} = \frac{-3}{1} = -3$

A. ∞ B. $-\infty$ C. 0 D. $\frac{4}{7}$ E. -3

Rich People Rule

22. $\lim_{x\to\infty} \frac{-3x^2+7x^3+2}{2x^3-3x^2+5} = \lim_{x\to\infty} \frac{7\cancel{x^3}}{2\cancel{x^3}} = \frac{7}{2}$

A. ∞ B. $-\infty$ C. 1 D. $\frac{7}{2}$ E. $-\frac{3}{2}$

Rich People Rule

25. $\lim_{x\to-\infty} \frac{2-5x}{\sqrt{x^2+2}} = \lim_{x\to-\infty} \frac{-5x}{|x|} = \frac{-5(-RBN)}{|-RBN(same)|} = \frac{5 \cdot \cancel{RBN}}{\cancel{RBN}} = 5$

A. 5 B. -5 C. 0 D. $-\infty$ E. ∞

$\sqrt{x^2} = |x|$

$\lim_{x\to 3^-} H(x) = \lim_{x\to 3^-} 3x-5 = 3\cdot 3-5 = 9-5 = 4$

27. Consider the function $H(x) = \begin{cases} 3x-5, & x<3 \\ x^2-2x, & x\geq 3 \end{cases}$. Which of the following statements is/are true?

I. $\lim_{x\to 3^-} H(x) = 4$.
True

II. $\lim_{x\to 3} H(x)$ exists.
False

III. $H(x)$ is continuous at $x=3$.
False
Limit does not Exist

A. I only B. II only C. I and II only

D. I, II and III E. None of these statements is true

$\lim_{x\to 3^-} H(x) = 4$

$\lim_{x\to 3} H(x) = \emptyset$ = Not The Same

$\lim_{x\to 3^+} x^2-2x = 3^2-2(3) = 9-6 = 3$

26. The function $f(x)=\dfrac{2x^2+x-3}{x^2+4x-5}$ has a vertical asymptote at $x=-5$ because…

$(x+5)(x-1)$

A. $\lim_{x\to -5^+} f(x)=\infty$ False

B. $\lim_{x\to -5^-} f(x)=-\infty$ False

C. $\lim_{x\to -5^-} f(x)=\infty$ True

D. $\lim_{x\to \infty} f(x)=-5$ — False ?

hole at $x=1$

E. $f(x)$ does not have a vertical asymptote at $x=-5$

$\lim_{x\to -5} \dfrac{\cancel{(x-1)}(2x+3)}{(x+5)\cancel{(x-1)}} = \dfrac{-7}{\pm RSN} = \pm\infty$

$2x^2+x-3$

$2\cdot -3=-6$

$3\cdot -2=1$

$2x^2+3x-2x-3$

$x(2x+3)-1(2x+3)$

$(x-1)(2x+3)$

$-5.01+5=-.01$

$-4.99+5=.01$

$\lim_{x\to -5^+} \dfrac{2x+3}{x+5}=\dfrac{-7}{+RSN}=-\infty$

$\lim_{x\to -5^-} \dfrac{2x+3}{x+5}=\dfrac{-7}{-RSN}=+\infty$

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8. $\lim_{x\to 3^+} \dfrac{x^2-4}{x^2-x-6} = \lim_{x\to 3^+} \dfrac{x^2-4}{x^2-x-6} = \lim_{x\to 3^+} \dfrac{(x-2)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}} = \dfrac{1}{+RSN} = +\infty$

DNE

$3.01-2=1.01$

$3.01-3=.01$

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13. For what value(s) of a is g(x) continuous at $x=2$?

$\cos\pi=-1$

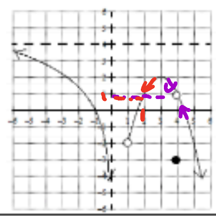
$a(2)^2+2(2)=-1$

$4a+4=-1 \Rightarrow 4a=-5$

$a=-\dfrac{5}{4}$

For question 12 – 16, use the equation $g(x)$ below and the graph of the function $f(x)$.

Graph of $f(x)$



$$g(x)=\begin{cases}3|x+3|, & x<-2\\ \cos\left(\frac{\pi x}{2}\right), & -2\le x<2\\ ax^2+2x, & x\ge 2\end{cases}$$

$a=-\frac{5}{4}$ $\lim_{x\to 2^+}$

#13 $a=-5/4$

No.

12. Is $g(x)$ is continuous at $x=-2$. [Base your response on the three part definition of continuity.]

$\lim_{x\to -2} g(x) = DNE$

$\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+} \cos\frac{\pi x}{2} = \cos\frac{-2\pi}{2} = \cos -\pi = -1$

Not The Same

$\lim_{x\to -2^-} g(x) = \lim_{x\to -2^-} 3|x+3| = 3|-2+3| = 3\cdot|+1| = 3$

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14. For what value(s) of b is the function $f(x)$ discontinuous? At which of these values does $\lim_{x\to b} f(x)$ exist? Explain your reasoning.

$x=0$

$[0,1]$

$x=4$

Limit Exists

$\lim_{x\to 4} F(x)=1$

$\lim_{x\to 0} F(x) = DNE = -\infty$

$\lim_{x\to 1} F(x) = DNE$

$\lim_{x\to 1^+} F(x) = -2$

$\lim_{x\to 1^-} F(x) = DNE$

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15. Find $\lim_{x\to 2^+}[g(x)+2f(x)]$.

$\lim_{x\to 2} F(x)=1$

$-5+4=-1$

$= -1+2(1)$

$\lim_{x\to 2^+} g(x) = \lim_{x\to 2} -\frac{5}{4}(x)^2+2x = -\frac{5}{\cancel{4}}(\cancel{4})+2\cdot 2 = -1$

6. $\lim_{x\to 0} \frac{x^3 - 2x^2 + 3x}{x}$

$$\lim_{x\to 0} \frac{\cancel{x}(x^2-2x+3)}{\cancel{x}} = 0^2 - 2(0) + 3 = 3$$

7. $\lim_{x\to 0} \frac{3\tan x}{x\sec x}$

$$\lim_{x\to 0} \frac{\frac{3\cdot \sin x}{\cos x}}{\frac{x}{\cos x}} = \lim_{x\to 0} \frac{3\sin x}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{x}$$

$$\lim_{x\to 0} 3\boxed{\frac{\sin x}{x}} = 3\cdot 1 = 3$$

11. $\lim_{x\to\infty} 5 + \frac{5}{x} = 5 + \frac{5}{RBN} = 5 + 0 = 5$

Rich Person Rule

$$\lim_{x\to s} \frac{x-s}{\sqrt{x}-\sqrt{s}} =$$

$$\lim_{x\to s} \frac{(\sqrt{x})^2 - (\sqrt{s})^2}{\sqrt{x}-\sqrt{s}}$$

$$\lim_{x\to s} \frac{\cancel{(\sqrt{x}-\sqrt{s})}(\sqrt{x}+\sqrt{s})}{\cancel{(\sqrt{x}-\sqrt{s})}}$$

$$\sqrt{s}+\sqrt{s} = 2\sqrt{s}$$

$$\lim_{x\to s} \frac{x-s}{\sqrt{x}-\sqrt{s}} =$$

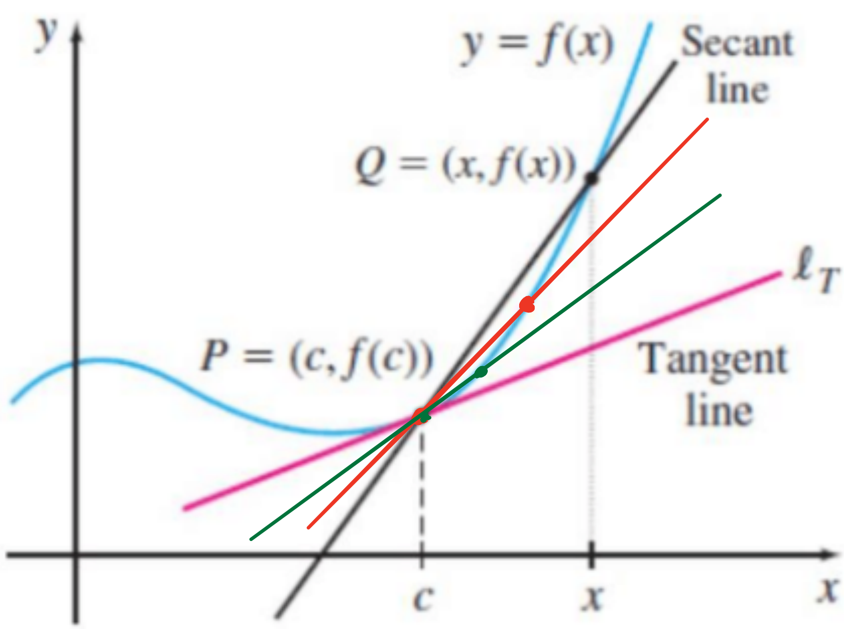
$$\lim_{x\to s} \frac{(x-s)}{(\sqrt{x}-\sqrt{s})} \frac{(\sqrt{x}+\sqrt{s})}{(\sqrt{x}+\sqrt{s})} = \lim_{x\to s} \frac{(x-s)(\sqrt{x}+\sqrt{s})}{x - \cancel{\sqrt{sx}} + \cancel{\sqrt{sx}} - s}$$

$$\frac{s-s}{\sqrt{s}-\sqrt{s}} = \frac{0}{0}$$

$$\lim_{x\to s} \frac{\cancel{(x-s)}(\sqrt{x}+\sqrt{s})}{\cancel{(x-s)}} = \sqrt{s}+\sqrt{s} = 2\sqrt{s}$$

$\sqrt{x^2} = |x| \Rightarrow$ $x = -3$ $\sqrt{(-3)^2} = \sqrt{9} = 3$

$(\sqrt{x})^2 = x \Rightarrow (\sqrt{-3})^2 = (i\sqrt{3})^2 = i^2 \cdot (\sqrt{3})^2 = -1 \cdot 3 = -3$



IRC

$f(x) = x^2$ at $x = -2$.

$F(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$\lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - [\cancel{x^2}]}{h}$

$\lim_{h \to 0} \frac{\cancel{h}(+2x + h)}{\cancel{h}} = 2x$

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = F'(x) = 2x$

$F'(-2) = 2(-2) = -4$

Example 4:

slope between 2 points

Find the average rate of change of the function $f(x) = x^2 - 4$ on the interval [1,3]:

$F(1) = 1^2 - 4 = -3$

$F(3) = 3^2 - 4 = 5$

$(1,-3)$
$(3,5)$ $= \frac{-3-5}{1-3} = \frac{-8}{-2} = 4$

Example 7:

Find the derivative of $f(x) = \frac{1}{x}$. Find $f'(1)$.

$c =$ ______

$F'(x)$

$F(x+h) = \frac{1}{x+h}$

$$\lim_{h \to 0} \frac{\frac{1 \cdot x}{(x+h) \cdot x} - \frac{1(x+h)}{x(x+h)}}{h}$$

$$\lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} = \lim_{h \to 0} \frac{\frac{\cancel{x} - \cancel{x} - h}{x(x+h)}}{\frac{h}{1}}$$

$$\lim_{h \to 0} \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} \Rightarrow F'(x) = \frac{-1}{x \cdot x} = \frac{-1}{x^2} =$$

$$F'(1) = \frac{-1}{1^2} = -1$$